The use of Homotopy Analysis Method for Indirect Trajectory Optimization

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This study develops an indirect optimal control solver based on the Homotopy Analysis Method. A finite interval boundary value problem is generated by using the necessary conditions of optimality and applying the Pontryagin’s Minimum Principle. Validation of the solver is performed using comparisons with a collocation-based boundary value problem solver, MATLAB’s bvp4c. The results demonstrate how simple initial guess to the boundary value problem can be used to produce high quality solutions. The accuracy of the solutions can be increased by increasing the order of solution at the cost of computational resources.

Nomenclature

\begin{itemize}
  \item \( N \) Non-linear governing equation
  \item \( B \) Linear boundary conditions
  \item \( L \) Linear operator
  \item \( H_a \) Auxiliary function
  \item \( c_o \) Convergence control parameter
  \item \( \delta \) Homotopy-derivative operator
  \item \( \phi, \psi \) Homotopy-Maclaurin series
  \item \( e_i \) Basis functions
  \item \( x, y \) State variables
  \item \( u, w \) State variables
  \item \( r \) Spatial variable
  \item \( E_{md} \) Discrete squared residual
  \item \( q \) Embedding parameter
  \item \( \lambda \) Co-state variable
  \item \( J \) Cost functional
  \item \( R \) Horizontal range, km
  \item \( h \) Altitude, km
  \item \( V_x \) Horizontal velocity, km/s
  \item \( V_y \) Vertical velocity, km/s
  \item \( g \) Acceleration due to Earth’s gravity, m/s^2
  \item \( V \) Boat speed, m/s
  \item \( \theta, \alpha \) Steering angle, deg
\end{itemize}

Subscript

\begin{itemize}
  \item \( m \) Order of HAM series solution
  \item \( n \) Order of governing differential equation
\end{itemize}

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I. Introduction

Indirect methods for trajectory optimization are based on using Calculus of Variation to solve for the first-order necessary conditions for optimality. The problem is then converted to a boundary value problem (BVP) which is further solved to give optimal trajectories. Indirect methods produce highly accurate trajectories which makes them very popular in the aerospace industry. However, these methods in general suffer from a number of issues which can be summarized as:

A narrow region of convergence due to local convergence properties and numerical instabilities of the problem resulting in the requirement of a very good initial guess. Sometimes the problem is hypersensitive and achieving convergence becomes almost impossible.

Popular indirect methods include the single shooting, multiple-shooting and the collocation methods. For the current study, trajectory optimization problems are solved using two solvers- the Homotopy Analysis Method based solver and the collocation based MATLAB’s \textit{bvp4c} function. \textit{bvp4c} divides the time interval into subintervals and discretizes the differential equations along the intervals. The non-linear system of equations resulting from the boundary conditions and the differential equations is solved using the Newton iteration method. However, a very good initial guess for the co-states is often required for convergence of the solution.

Highly nonlinear initial value and boundary value problems can also be solved using analytical approximation methods. The Homotopy Analysis Method is an analytic approximation method, which has gained popularity in solving initial value problems arising in science, finance, and engineering after it was proposed by Dr. Shijun Liao in 1992. It provides us with the benefit of controlling the convergence region of the problem. It is independent of any parameter and provides great flexibility in the choice of initial guess. Liao demonstrated the validity of Homotopy Analysis Method by solving the differential equations resulting from some highly non-linear problems. By solving the Blasius flow equation using Homotopy Analysis Method, Liao showed an increase in the size of the convergence region as compared to that of the original Blasius power series solution, which makes HAM interesting enough to be investigated for the use in trajectory optimization problems.

HAM has been used to solve linear and non-linear optimal control problems (OCPs). Zahedi and Nik applied the original HAM approach to solve linear OCPs with quadratic performance index. In another study, HAM was applied to solve a non-linear OCP to find the optimal maneuvers of a rigid asymmetric spacecraft and compared the results generated using the \textit{bvp4c} function. However, most of the OCPs solved in the past are based on linear equations of dynamics. The current study shows how HAM can be applied to indirect trajectory optimization problems with non-linear dynamics in improving the convergence properties by reducing the effort to provide the initial guess. A generalized approach to provide initial guess for the HAM based indirect method makes it more reliable for trajectory optimization.

II. Homotopy Analysis Method Theory & Background

To understand the basic idea of HAM, as developed by Dr. Shijun Liao, we consider an Initial Value Problem (IVP) and extend the discussion to a BVP. Let one of the governing equations be given by a \textit{n}th order non-linear ODE

\[ N[u(r, t), t] = 0, t \in [0, T] \]  \hspace{1cm} (1)

subject to \text{n} linear initial boundary conditions,

\[ B_k[t, u] = \gamma_k, 1 \leq k \leq n \]  \hspace{1cm} (2)

where, \( N \) is a \textit{n}th order differential operator, \( B_k \) is a linear operator, \( u(t) \) is a smooth function, \( t \) is a temporal variable, \( r \) is a spatial variable, and \( T \geq 0 \). For each governing equation \( N \), using an embedding parameter \( q \), Liao suggested to construct a zeroth-order homotopy deformation equation given by Eq. (3), so that the Homotopy-Maclaurin series solution for \( N \), given by \( \phi(r, t; q) \) exists and is analytic at \( q=0 \). The Homotopy-Maclaurin series solution is shown in Eq. (4).

\[ (1 - q)L[\phi(r, t; q) - u_0(r, t)] = c_oqH_N[t, \phi(r, t; q)] \hspace{1cm} c_o \neq 0 \]  \hspace{1cm} (3)
\[ \phi(r, t; q) = u_0(r, t) + \sum_{m=1}^{\infty} u_m(r, t)q^m, q \in [0, 1] \]  

\[ L \] and \( u_0(r, t) \) are the linear operator and the initial guess respectively, both provided by the designer. \( m \) is the order of Maclaurin series, \( c_o \) is an auxiliary convergence control parameter, \( H_a \) is a non-zero auxiliary function provided by the designer, and \( u_m \) is given by Eq. (5).

From Eq. (3), it can be observed that, as \( q \) increases from 0 to 1, the Homotopy-Maclaurin series \( \phi(r, t; q) \) varies continuously from \( u_0(r, t) \) to \( u(r, t) \). We assume that the linear operator, initial guess, auxiliary function and \( c_o \) are chosen such that the Maclaurin series converges at \( q = 1 \). We differentiate the zeroth order deformation equation, Eq. (3) \( m \) times with respect to \( q \) and divide it by \( m! \) to obtain the \( m \)th order deformation equation given by Eq. (6). Integrating Eq. (6) with the linear boundary conditions, Eq. (7) gives the HAM series solution, \( u(r, t) \) [Eq. (8)]. The value of \( \chi_m \) is given by Eq. (9).

\[ L[u_m(r, t) - \chi_m u_{m-1}(r, t)] = H_a c_o \delta_{m-1}(N[t, \phi(r, t; q)]) \]  

\[ u_m(r, 0) = 0 \]  

\[ u(r, t) = u_0(r, t) + \sum_{m=1}^{\infty} u_m(r, t) \]  

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1 \end{cases} \]  

\[ \delta_k(\phi) \] is called as the \( k \)th-order homotopy-derivative operator given in Eq. (10).

\[ \delta_k(\phi) = \left( \frac{1}{k!} \frac{\partial^k \phi}{\partial q^k} \right) \]  

In practice, the series solution obtained by Eq. (8) is truncated to a finite number of terms, which gives us the \( M \)th order approximation as

\[ u(r, t) = u_0(r, t) + \sum_{m=1}^{M} u_m(r, t) \]  

For each value of \( m = 1, 2, 3, \ldots \), Eq. (6) converts the non-linear ODE, \( N[u(r, t), t] = 0 \) to a linear ODE with order equal to the order of the linear operator in use.

**II.A. HAM Approach for a Boundary Value Problem**

For solving a BVP using HAM, it is first formulated as an IVP. The known initial boundary conditions from Eq. (2) are used in formulating the initial guess as explained below. For the state variables with unknown initial boundary conditions, we assume their values to be finite parameters \( \beta_1, \beta_2, \ldots, \beta_n \). After obtaining the series solutions for each state and co-state variable, we use the given terminal boundary conditions to root-solve a non-linear system of \( n \) equations to obtain the values of the finite parameters \( \beta_1, \beta_2, \ldots, \beta_n \).

The approach is demonstrated with the help of a simple optimal control problem as follows. The objective functional is defined as

\[ \text{Min } J = \int_0^1 (x^2 + u^2)dt \]  

with the dynamics

\[ \dot{x} = u, \ x(0) = 1, \ t \in [0, 1] \]
where $u$ is the control variable. On applying the Euler-Lagrange theorem, we obtain the following BVP,

$$\dot{x} = -\lambda, \quad \dot{\lambda} = -x, \quad x(0) = 1, \quad \lambda(1) = 0$$  \hspace{1cm} (14)

where $\lambda$ is a co-state. This problem has an analytical solution given by Eq. (15)

$$x(t) = \frac{e^t + e^2e^{-t}}{1 + e^2}, \quad \lambda(t) = -\frac{e^t - e^2e^{-t}}{1 + e^2}$$  \hspace{1cm} (15)

Homotopy-Maclaurin series for the state and co-state are defined in the following Eq. (16).

$$\phi(t; q) = x_0(t) + \sum_{m=1}^{+\infty} x_m(t)q^m, \quad q \in [0, 1]$$

$$\psi(t; q) = \lambda_0(t) + \sum_{m=1}^{+\infty} \lambda_m(t)q^m, \quad q \in [0, 1]$$  \hspace{1cm} (16)

where $x_m$ and $\lambda_m$ can be obtained by integrating the respective $m^{th}$ order deformation equations given by Eq. (17).

$$L[x_m(t) - \chi_m x_{m-1}(t)] = H_a c_o \delta_{m-1}(N_1[t, \phi(t; q)])$$

$$L[\lambda_m(t) - \chi_m \lambda_{m-1}(t)] = H_a c_o \delta_{m-1}(N_2[t, \psi(t; q)])$$  \hspace{1cm} (17)

subject to the boundary conditions

$$x_m(0) = 0$$

$$\lambda_m(0) = 0$$  \hspace{1cm} (18)

where

$$N_1 : \dot{\phi} + \psi = 0$$

$$N_2 : \dot{\psi} + \dot{\phi} = 0$$  \hspace{1cm} (19)

$M^{th}$ order HAM solution for the state and co-state variables is represented as

$$x(t) = x_0(t) + \sum_{m=1}^{M} x_m(t)$$

$$\lambda(t) = \lambda_0(t) + \sum_{m=1}^{M} \lambda_m(t)$$  \hspace{1cm} (20)

### II.B. Selection of Initial Guess, Linear operator, & Auxiliary Function

Although, there are no conclusive proofs and rigorous theories to select the initial guess, linear operator, and the auxiliary function, HAM theory provides suggestions for their selection to solve practical problems. Liao suggested to define a set of basis functions to correctly represent the series solution of the Eq. (1). A typical HAM series solution can be represented as a power series given by

$$u(t) = \sum_{m=1}^{+\infty} a_m c_m(t)$$  \hspace{1cm} (21)

where $a_m$ are the coefficients obtained by the HAM series solution, and $c_m(t)$ are the basis functions, chosen by the designer to represent the series solution. Eq. (21) is called as the rule of solution expression. The initial guess, linear operator, and the auxiliary function has to be chosen in such a way so that they satisfy the rule of solution expression as explained below.
II.B.1. Initial Guess

The initial guess must be chosen such that it can be expressed by the sum of the basis functions defined above. Also, the initial guess for a state must satisfy the maximum possible number of boundary conditions for that state. Eq. (22) shows a typical representation of the initial guess for a state

\[ x_0(t) = \sum_{m=1}^{n} b_m e_m(t) \]  

where \( n \) is the number of boundary conditions on the state, \( b_m \) are the finite coefficients chosen by the designer to satisfy the boundary conditions, and \( e_m(t) \) are the basis functions chosen to represent the series solution.

II.B.2. Linear Operator

The linear operator must be chosen such that the solution of Eq. (23) is expressed as the sum of the basis functions chosen earlier and is given by Eq. (24)

\[ L[w(t)] = 0 \]  

\[ w(t) = \sum_{m=1}^{K_1} d_m e_m(t) \]  

where, \( d_m \) are a set of finite coefficients chosen by the designer and \( K_1 \) is a positive integer. There is no strict rule to select the value of \( K_1 \), but it is suggested that in most of the problems, it should be chosen as the highest order of derivative of the original Eq. (1).

For finite time interval BVPs, where \( t \in [0, T] \), \( H_a \) is simply used as 1. For the current problem, we use the simplest rule of solution expression, a polynomial power series given by Eq. (26) for which the set of basis functions is the following set \( e_m \)

\[ e_m(t) = [1, t, t^2, t^3, ...] \]  

\[ x(t) = a_1 + a_2 t + a_3 t^2 + ... \]  

where \( a_1, a_2, \) and \( a_3 \) are the coefficients which are obtained by the HAM series solution. In general, we have the freedom to choose the basis functions as polynomials, trigonometric functions, Fourier series or a combination of them.

The approach mentioned in Section II.B.1 is used to select the initial guesses for the state and co-state. For convenience, we decide to select the initial guesses which satisfy only the initial boundary conditions for both the state and the co-state. This assumption results in \( n = 1 \) for the initial guess. Hence,

\[ x_0(t) = \sum_{m=1}^{1} b_{1m} e_m(t) = b_{11} e_1 \]  

\[ \lambda_0(t) = \sum_{m=1}^{1} b_{2m} e_m(t) = b_{21} e_1 \]  

To satisfy the initial boundary condition on the state, we select \( b_{11} = 1 \). Since the initial boundary condition for the co-state is unknown, we assume it to be some finite value \( \beta_1 \) as explained in Section II.A. We select \( b_{21} = \beta_1 \) to satisfy the initial boundary condition on the co-state. The initial guesses for the state and co-state are now calculated to be:

\[ x_0(t) = 1 \]  

\[ \lambda_0(t) = \beta_1 \]  

For the linear operator, we use the approach described in Section II.B.2 to define \( w(t) \) as shown in Eq. (29). Since the highest order derivative for both the original governing equations is 1, the value for \( K_1 \) is chosen...
to be 1. We can now define \( w(t) \) as

\[
w(t) = \sum_{m=1}^{1} d_m c_m = d_1 e_1 = d_1
\]

where, \( d_1 \) is a constant. The linear operator is chosen to be a first order derivative, such that it satisfies Eq. (23) as follows:

\[
L(w(t)) = \frac{d}{dt}(d_1) = 0
\]

Using Eq. (17) and Eq. (19), the \( m^{th} \) order deformation equations can now be written for the state and costate as

\[
x_m(t; c_o) = \chi_m x_{m-1}(t) + c_o \int_0^t H_a \delta_{m-1}(\dot{x} + \lambda) dt + C_1
\]

\[
\lambda_m(t, c_o) = \chi_m \lambda_{m-1}(t) + c_o \int_0^t H_a \delta_{m-1}(\dot{\lambda} + x) dt + C_2
\]

\( C_1 \) and \( C_2 \) are constants of integration determined by the initial conditions given by Eq. (18). As mentioned previously, we assume \( H_a = 1 \).

II.C. Auxiliary Convergence Control Parameter \((c_o)\)

HAM guarantees the convergence of the series solution,\(^{12}\) which counts as one of the major advantages to solve BVPs. MATLAB’s symbolic toolbox is used to solve Eq. (31). We obtain terms for \( x_m \) and \( \lambda_m \) and substitute them into Eq. (20), to obtain a family of series solutions in \( c_o \). The solutions for state and co-state are functions of the independent variable \( t \) and \( c_o \). The Homotopy Analysis Method provides us the freedom to choose the value of \( c_o \) to adjust the region and the rate of convergence. Liao suggested to plot the curves of physical quantities like \( \dot{c} \) and \( \ddot{c} \) are functions of the independent variable \( t \) to study their dependency on \( c_o \), where \( t' \) can be any instant of time in the domain of the problem. These curves are termed as \( c_o \sim \) curves and are denoted as \( \dot{c} \sim c_o \) and \( \ddot{c} \sim c_o \) for any state or physical quantity. According to the HAM convergence theorem,\(^{12}\) all convergent series of \( \dot{c} \) and \( \ddot{c} \) converge to constant values for a specific range \((R_{c_o})\) of \( c_o \) values, resulting in a horizontal line in the \( c_o \sim \) curves. Regardless of the initial guess, and for any value of \( c_o \) in \( R_{c_o} \), the same value of the physical quantity is obtained, and the series solution is said to converge.

II.D. Discrete Squared Residual \((E_{md})\)

Squared residual is defined as a measure of how well the power series satisfies the governing equations integrated over the whole domain. The squared residual for any governing equation is calculated as

\[
E_m(c_o) = \int_0^T N\{\sum_{n=1}^{m} u_m(t, c_o)\}^2 dt
\]

where \( T \) is the final value of time interval used in Eq. (1), \( E_m \) is the squared residual for the governing equation, obtained at the \( m^{th} \) order series solution. \( c_o \) plays an important role in determining the residual for any series solution. As proposed by Liao,\(^{11}\) once \( R_{c_o} \) is determined, the optimal value of \( c_o \) can be calculated by minimizing the squared residual within \( R_{c_o} \).

Due to the high computational requirements for \( E_m \), a discrete squared residual \( E_{md} \) is also defined for the \( m^{th} \) order series solution as

\[
E_{md} = \frac{1}{N_{step} + 1} \sum_{j=0}^{N_{step}} \left\{ \Delta_m(\tau_j; c_o) \right\}^2, \tau_j = \frac{t_f j}{N_{step}}
\]

where

\[
\Delta_m(\tau; c_o) = N(u_m(\tau; c_o))
\]

\( N_{step} \) is the number of time steps used, and \( t_f \) is the final time of the OCP. For the current study, \( N_{step} \) is assumed to be 40. An overall discrete squared residual \( E_{md,total} \) can be defined by adding the discrete squared residuals for each governing equation as follows

\[
E_{md,Total} = E_{md,N_1} + E_{md,N_2} + E_{md,N_3} + ...
\]
where $E_{md,N_1}$ is the discrete squared residual for the governing equation $N_1$.

The $m^{th}$ order deformation Eq. (31) is solved to obtain the analytical terms for $x_m$ and $\lambda_m$ terms in $c_o$ and $\beta_1$. First, the value of $c_o$ is assumed to be -1. Then, the final boundary condition on the co-state is used to obtain a non-linear equation which is root solved for $\beta_1$. Using $\beta_1$, the $M^{th}$ order series solutions for both the state and co-state are evaluated.

Further, the $c_o \sim$ curves are plotted to understand the convergence properties of the state and co-state series solutions. For this problem, the quantities $x \sim c_o$, $\dot{x} \sim c_o$, $\ddot{x} \sim c_o$ for the state and $\lambda \sim c_o$, $\dot{\lambda} \sim c_o$, $\ddot{\lambda} \sim c_o$ for the co-state are used. Theoretically, since the curves converge at each instant of time, the final time of 1 s is chosen arbitrarily. Figs. 1(a) and 1(b) show the $c_o \sim$ curves for the $3^{rd}$ order and $5^{th}$ order series solutions respectively. The two plots can be compared to conclude that the convergence region increases with an increase in the order of solution, giving designers more freedom in choice of $c_o$. For the $5^{th}$ order solution, a common $c_o$ range of $[-1.2, 0]$ could be identified for both the state and the co-state in which the curves converge to constant values for all the mentioned physical quantities.

$E_{md,Total}$ given by Eq. (35), is minimized to obtain the optimal $c_o$ for the $5^{th}$ order solution. MATLAB’s $fminbnd$ function based on the Golden Section Search Algorithm with parabolic interpolation is used to minimize $E_{md,Total}$ for the range $[-1.2, 0]$. The optimal value of $c_o$ for the $5^{th}$ order solution is found to be -0.9567. The optimal $c_o$ is then used to obtain the updated series solutions for the state and co-state. $\beta_1$ is root solved again by using the terminal boundary condition on the co-state. This method of using an updated value of $c_o$ to obtain the series solutions is known as “convergence control”.

Table 1 shows a reduction in the total discrete squared residual obtained by using an optimal value of $c_o$, as compared to $c_o$ of -1. The computations were performed on an Intel(R) Xeon(R) CPU E3-1225 v3 3.20 Ghz (4 CPUs) processor. Fig. 2 shows the performance curve for the problem. Reduction in the discrete squared residual is traded with increase in the computational cost. It is found that the residual decreases with increase in the order of series solution.

Table 1: Effect of $c_o$ on $E_{md,Total}$ - Simple Optimal Control Problem

<table>
<thead>
<tr>
<th>$c_o$</th>
<th>$E_{md,Total}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$3.17 \times 10^{-5}$</td>
</tr>
<tr>
<td>-0.9567</td>
<td>$1.27 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
The series solutions for both the state and costate are compared with the analytical solution in the Fig. 3. The series solutions without convergence control \((c_o = -1)\) are also compared with the solutions obtained using the optimal \(c_o\). Although, both of the values of \(c_o\) lie in the convergence region \([R_{c_o}]\), a small improvement is obtained by using the optimal value of \(c_o\). The 5\(^{th}\) order series solution is given by Eq. (36).

\[
x(t) = 1 - 0.76t + 0.5t^2 - 0.12t^3 + 0.04t^4 - 0.01t^5
\]
\[
\lambda(t) = 0.76 - t + 0.38t^2 - 0.16t^3 + 0.03t^4 - 0.01t^5
\]

(36)

Different initial guesses were used to compute the HAM series solution. Table 2 shows the \(E_{md,Total}\) values obtained at 5\(^{th}\) order HAM solution for the initial guesses used. The initial guess given in the last row is based on an exponential rule of solution expression. Since the analytical solution of the problem (Eq. (15)) contains exponential terms, the exponential series rule of solution expression is an apt choice for this particular problem. This fact is also confirmed by the least value of \(E_{md,Total}\) obtained by using the exponential initial guess as compared to other expressions.
### Table 2: Effect of Initial Guess on $E_{md,Total}$

<table>
<thead>
<tr>
<th>Initial Guess $[x_0, \lambda_0]$</th>
<th>$E_{md,Total}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[1, \beta_1]$</td>
<td>$1.27 \times 10^{-6}$</td>
</tr>
<tr>
<td>$[1, \beta_1(1-t)]$</td>
<td>$1.90 \times 10^{-6}$</td>
</tr>
<tr>
<td>$[1, \beta_1 e^t]$</td>
<td>$1.37 \times 10^{-5}$</td>
</tr>
<tr>
<td>$[e^t, \beta_1 e^t]$</td>
<td>$1.41 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

### III. Zermelo’s Problem

Zermelo’s problem\(^{13}\) consists of minimizing the time required by a boat to cross a river. Fig. 4 shows a schematic of the optimal control problem. $\theta$ is the boat steering angle from the horizontal direction which is varied continuously to reach the terminal point across the river in the minimum possible time. The boat is assumed to move with a constant velocity, $V$ of 1 m/s.

![Figure 4: Schematic for Zermelo’s problem. Figure in citation.\(^{13}\)](image)

The objective function, $J$, is defined as

$$\text{Min } J = t_f$$

with the dynamics

$$\dot{x} = V \cos \theta$$

$$\dot{y} = V \sin \theta$$

where, the states $x$ and $y$ are the horizontal and vertical coordinates respectively. We use the Euler-Lagrange theorem to obtain the dynamics for the co-states given in Eq. (39). The control law is evaluated and is given by Eq. (40).

$$\dot{\lambda}_x = 0$$

$$\dot{\lambda}_y = 0$$

$$\tan \theta = \frac{\lambda_y}{\lambda_x}$$

The boat starts from $(0,0)$ and crosses the river to reach $(1,1)$. The set of boundary conditions along with the transversality condition obtained for the final time are given in Eq. (41).

$$x(0) = 0, \ y(0) = 0$$

$$x(t_f) = 1, \ y(t_f) = 1$$

$$H_{t_f} = -1$$

where $H_{t_f}$ is the value of the hamiltonian calculated at the final time $t_f$. This problem has an analytical solution given by Eq. (42).
The Homotopy-Maclaurin series and the $m^{th}$ order deformation equations for the states and co-states are formulated. A set of polynomial functions as the basis functions and a rule of solution expression similar to the one used in the previous problem given by Eq. (26) is chosen for Zermelo’s problem. Similar to the process before, for convenience, we select the initial guesses which satisfy only the initial boundary conditions on the states and co-states. The initial guess constructed is given in the Table 3.

Table 3: Initial Guess for Zermelo’s Problem

<table>
<thead>
<tr>
<th>State/Co-state</th>
<th>Initial Guess</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0</td>
</tr>
<tr>
<td>$y$</td>
<td>0</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>$\beta_1$</td>
</tr>
<tr>
<td>$\lambda_y$</td>
<td>$\beta_2$</td>
</tr>
</tbody>
</table>

Since the highest order of derivative for all the governing equations is 1, we select the value of $K_1$ defined in Eq. (24) to be 1. Similar to the previous problem, the linear operator is calculated to be $\frac{d}{dt}$. The auxiliary function is chosen to be 1. Since we are also required to calculate the optimal final time for the problem, another unknown parameter, $t_f$, is used. We start by selecting the value of $c_0$ to be -1. The terminal boundary conditions on the states and the transversality condition on the final time parameter, $t_f$, are applied to formulate a non-linear system of equations in the parameters $\beta_1$, $\beta_2$, and $t_f$. MATLAB’s `Fsolve` function is used to numerically solve the non-linear system of equations.

### III.B. Results for Zermelo’s Problem

Fig. 5 shows the $c_0 \sim$ curves for both the state variables at the $5^{th}$ order solution for $c_0 \in [-2, 0]$. Since the co-state values are constant, they are not dependent on $c_0$. For the states, $c_0 = -1$ lies in the horizontal convergence region, and can be used to evaluate the series solutions. Hence, “convergence control” is not required for this problem.
The series solutions for the states and co-states are compared with the analytical results in Fig. 6. State variable trajectories and $t_f$ calculated from the HAM approach agree with the analytical solution. Co-state variables are constant and can take any non-zero real value. Fig. 7(a) compares the control history obtained using the two methods. A constant steering angle of 45 deg is needed to be maintained for the boat to reach the terminal point. Fig. 7(b) shows that a first order HAM solution results in an $E_{md, Total}$ of the order of $10^{-26}$.

![Figure 6: State and Co-state - 5th order HAM solution for Zermelo’s Problem](image)

![Figure 7: Results - Zermelo’s Problem](image)

## IV. 2D Ascent Problem

A finite time 2D ascent problem is solved to demonstrate the applicability of the HAM based indirect method on a non-linear aerospace trajectory optimization problem. The 2D ascent problem is a popular optimal control problem emulating the satellite launch from the surface of the Earth to an orbit of fixed altitude in the minimum possible time.

### IV.A. Problem Formulation

A modified 2D ascent problem has been constructed to test the HAM approach on the optimal control problem. A flat Earth model is assumed as shown in Fig. 8. The objective of this optimal control problem is to maximize the final horizontal velocity of the vehicle in orbit in a given fixed time. The original 2D ascent problem has been simplified to a fixed final time problem by using the following assumptions:

1. A vehicle launched from the surface of the Earth must reach an orbit of 185.2 km in 485 seconds to achieve a maximum terminal horizontal component of velocity.

2. An instantaneous steering angle $\alpha$ is the only control variable.
3. The acceleration due to gravity from the Earth is assumed to be 9.8 m/s$^2$.
4. Thrust to weight ratio for the vehicle is 3.
5. A constant mass and a constant thrust force $F$ is assumed.
6. Final altitude to be achieved is 185.2 km.
7. There is no atmosphere and no aerodynamic forces on the vehicle.

The objective function for this case is defined by Eq. (43)

$$\text{Min } J = -v_x f$$

with the dynamics given by Eq. (44). The thrust acceleration for the vehicle is calculated using Eq. (45)

$$\begin{align*}
\dot{R} &= v_x \\
\dot{h} &= v_y \\
v_x' &= \frac{F}{m_o} \cos \alpha \\
v_y' &= \frac{F}{m_o} \sin \alpha - g
\end{align*}$$

$$\frac{F}{m_o} = \text{(Thrust to weight)}(g) = 29.4 \text{ m/s}^2$$

The boundary conditions on the states and co-states for the boundary value problem formulated are given in Table 4.

<table>
<thead>
<tr>
<th>State/Co-state</th>
<th>Initial Condition</th>
<th>Terminal Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>0 km</td>
<td>free</td>
</tr>
<tr>
<td>$h$</td>
<td>0 km</td>
<td>185.2 km</td>
</tr>
<tr>
<td>$v_x$</td>
<td>0 km/s</td>
<td>free</td>
</tr>
<tr>
<td>$v_y$</td>
<td>0 km/s</td>
<td>0 km/s</td>
</tr>
<tr>
<td>$\lambda_R$</td>
<td>free</td>
<td>0</td>
</tr>
<tr>
<td>$\lambda_h$</td>
<td>free</td>
<td>free</td>
</tr>
<tr>
<td>$\lambda_{v_x}$</td>
<td>free</td>
<td>-1</td>
</tr>
<tr>
<td>$\lambda_{v_y}$</td>
<td>free</td>
<td>free</td>
</tr>
</tbody>
</table>
IV.B. HAM Formulation

Following the similar approach as described in the Section II.A, we formulate the HAM problem by defining the Homotopy-Maclaurin series and the m\textsuperscript{th} order deformation equations for each state and co-state variable.

The choice of the basis functions and the rule of solution expression is the same as used for the simple control problem given by Eq. (25). Same approach is used for the selection of initial guess as used for the previous problems. Only the initial boundary conditions are considered to build the initial guess, as shown in the Table 5. From Table 4, it can be seen that the initial conditions for the co-states are unknown. Hence, we select parameters for those values.

### Table 5: Initial Guess - 2D Ascent Problem

<table>
<thead>
<tr>
<th>State and Co-state Initial Guess</th>
<th>R</th>
<th>h</th>
<th>v\textsubscript{x}</th>
<th>v\textsubscript{y}</th>
<th>(\lambda)\textsubscript{R}</th>
<th>(\lambda) (\beta)</th>
<th>(\lambda)\textsubscript{v}\textsubscript{x}</th>
<th>(\lambda)\textsubscript{v}\textsubscript{y}</th>
<th>(\beta)</th>
<th>(\beta)</th>
<th>(\beta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Guess</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\beta)\textsubscript{1}</td>
<td>(\beta)\textsubscript{2}</td>
<td>(\beta)\textsubscript{3}</td>
<td>(\beta)\textsubscript{4}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Same approach for the selection of linear operator and auxiliary function are used as described in the Section II.B. The value of \(K\textsubscript{1}\) is chosen to be 1, since the highest order of derivative for all the governing equations is 1. Using Eq. (23), we obtain the linear operator as \(\frac{d}{d\lambda}\). The auxiliary function is selected to be 1. The 4 terminal boundary conditions given in Table 4 are used to build a nonlinear system of equations in \(\beta\)\textsubscript{1}, \(\beta\)\textsubscript{2}, \(\beta\)\textsubscript{3} and \(\beta\)\textsubscript{4}. Table 6 lists the values of parameters obtained at each order of solution.

### Table 6: Parameter Values - 2D Ascent Problem

<table>
<thead>
<tr>
<th>Order of solution</th>
<th>(\beta)\textsubscript{1}</th>
<th>(\beta)\textsubscript{2}</th>
<th>(\beta)\textsubscript{3}</th>
<th>(\beta)\textsubscript{4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>-2.3230\times10^{-4}</td>
<td>-1</td>
<td>-0.4144</td>
</tr>
<tr>
<td>3</td>
<td>0.0008</td>
<td>-2.3230\times10^{-4}</td>
<td>-1</td>
<td>-0.5975</td>
</tr>
<tr>
<td>4</td>
<td>3.7368\times10^{-30}</td>
<td>-6.2122\times10^{-4}</td>
<td>-1</td>
<td>-0.5133</td>
</tr>
<tr>
<td>5</td>
<td>1.1183\times10^{-4}</td>
<td>-0.0011</td>
<td>-1</td>
<td>-0.6333</td>
</tr>
<tr>
<td>6</td>
<td>4.8850\times10^{-31}</td>
<td>-7.4772\times10^{-4}</td>
<td>-1</td>
<td>-0.5423</td>
</tr>
<tr>
<td>7</td>
<td>-7.0779\times10^{-29}</td>
<td>-8.3634\times10^{-4}</td>
<td>-1</td>
<td>-0.5611</td>
</tr>
<tr>
<td>8</td>
<td>-5.7842\times10^{-31}</td>
<td>-7.8445\times10^{-4}</td>
<td>-1</td>
<td>-0.5501</td>
</tr>
<tr>
<td>9</td>
<td>-4.4299\times10^{-29}</td>
<td>-8.0683\times10^{-4}</td>
<td>-1</td>
<td>-0.5548</td>
</tr>
<tr>
<td>10</td>
<td>-7.9876\times10^{-16}</td>
<td>-7.9460\times10^{-4}</td>
<td>-1</td>
<td>-0.5522</td>
</tr>
<tr>
<td>10\textsuperscript{th} order for optimal (c)\textsubscript{o}</td>
<td>6.9437\times10^{-29}</td>
<td>-0.0007</td>
<td>-1</td>
<td>-0.5522</td>
</tr>
</tbody>
</table>

IV.C. Results for 2D Ascent Problem

Fig. 9 shows the \(c\)\textsubscript{o} curves for the 10\textsuperscript{th} order series solutions. No common horizontal region can be found out for which all the physical quantity converge. In this case, optimal \(c\)\textsubscript{o} is calculated to be -0.9996 by simply minimizing \(E\textsubscript{md,Total}\) as a function of \(c\)\textsubscript{o} for real values. Figs. 10 and 11 compare the state and co-state series solution with the \textit{bvp4c} solution. It can be seen from Fig. (10) that an initial guess of 0 is deformed into a highly non-linear solution using this approach. Fig. 12 compares the the control history obtained from the two methods. Performance curve for the ascent problem is shown by Fig. 13. An irregular peak for 5\textsuperscript{th} order solution is due to inability of the MATLAB’s numerical solver to obtain the solution in 2,000,000 function evaluations. Since the number of analytical terms in the deformation equations increase with the order, the CPU time for the 10\textsuperscript{th} order solution almost triples to that of the 9\textsuperscript{th} order solution.
Figure 9: $c_o \sim$ curves for $c_o \in [-2, 0]$ - 2D Ascent Problem

Figure 10: States for 10th Order HAM Solution - 2D Ascent Problem

Figure 11: Co-states for 10th Order HAM solution - 2D Ascent Problem
V. Conclusion and Future Work

HAM based indirect method for trajectory optimization is demonstrated on linear and non-linear optimal control problems. The HAM based solver exploits the flexibility and ease available for the initial guess as compared to the collocation solver. Use of convergence control in a region is also demonstrated in the approach. The HAM solver currently suffers from high computational times which can be further worked upon. This study concludes by showing an immense potential in the development of HAM based indirect solvers for trajectory optimization. Further work in this area is mentioned as follows

1. Process of integrating the symbolic deformation equations can be parallelized to reduce computational time.

2. Quantifying the rate of convergence can be done to observe the process of convergence control. This will further help in classifying optimal control problems based on their convergence properties.

3. As suggested in the theory, multiple convergence control parameters can be used further to control the rate of convergence.

4. For higher fidelity solutions, the approach can be used to build hybrid solvers with faster solvers like collocation and pseudospectral methods.

References


