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The use of Homotopy Analysis Method for Indirect Trajectory Optimization

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This study develops an indirect optimal control solver based on the Homotopy Analysis Method. A finite interval boundary value problem is generated by using the necessary conditions of optimality and applying the Pontryagin's Minimum Principle. Validation of the solver is performed using comparisons with a collocation-based boundary value problem solver, MATLAB's *bvp4c*. The results demonstrate how simple initial guess to the boundary value problem can be used to produce high quality solutions. The accuracy of the solutions can be increased by increasing the order of solution at the cost of computational resources.

Nomenclature

N	Non-linear governing equation
B	Linear boundary conditions
L	Linear operator
H_a	Auxiliary function
c_o	Convergence control parameter
δ	Homotopy-derivative operator
ϕ, ψ	Homotopy-Maclaurin series
e_i	Basis functions
x, y	State variables
u, w	State variables
r	Spatial variable
E_{md}	Discrete squared residual
q	Embedding parameter
λ	Co-state variable
J	Cost functional
R	Horizontal range, km
h	Altitude, km
V_x	Horizontal velocity, km/s
V_y	Vertical velocity, km/s
g	Acceleration due to Earth's gravity, m/s^2
V	Boat speed, m/s
θ, α	Steering angle, deg

Subscript

m	Order of HAM series solution
n	Order of governing differential equation

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I. Introduction

Indirect methods for trajectory optimization are based on using Calculus of Variation to solve for the first-order necessary conditions for optimality. The problem is then converted to a boundary value problem (BVP) which is further solved to give optimal trajectories. Indirect methods produce highly accurate trajectories which makes them very popular in the aerospace industry.[1] However, these methods in general suffer from a number of issues which can be summarized as:
A narrow region of convergence due to local convergence properties and numerical instabilities of the problem resulting in the requirement of a very good initial guess. Sometimes the problem is hypersensitive and achieving convergence becomes almost impossible.
Popular indirect methods include the single shooting,[2] multiple-shooting[3] and the collocation methods. For the current study, trajectory optimization problems are solved using two solvers- the Homotopy Analysis Method based solver and the collocation based MATLAB's *bvp4c*[4] function. *bvp4c* divides the time interval into subintervals and discretizes the differential equations along the intervals. The non-linear system of equations resulting from the boundary conditions and the differential equations is solved using the Newton iteration method. However, a very good initial guess for the co-states is often required for convergence of the solution.

Highly nonlinear initial value and boundary value problems can also be solved using analytical approximation methods. The Homotopy Analysis Method is an analytic approximation method, which has gained popularity in solving initial value problems arising in science, finance, and engineering after it was proposed by Dr. Shijun Liao in 1992.[5] It provides us with the benefit of controlling the convergence region of the problem. It is independent of any parameter and provides great flexibility in the choice of initial guess. Liao demonstrated the validity of Homotopy Analysis Method by solving the differential equations resulting from some highly non-linear problems. By solving the Blasius flow equation using Homotopy Analysis Method, Liao showed an increase in the size of the convergence region as compared to that of the original Blasius power series solution,[8] which makes HAM interesting enough to be investigated for the use in trajectory optimization problems.

HAM has been used to solve linear and non-linear optimal control problems (OCPs). Zahedi and Nik applied the original HAM approach to solve linear OCPs with quadratic performance index.[9] In another study, HAM was applied to solve a non-linear OCP to find the optimal maneuvers of a rigid asymmetric spacecraft and compared the results generated using the *bvp4c* function.[10] However, most of the OCPs solved in the past are based on linear equations of dynamics. The current study shows how HAM can be applied to indirect trajectory optimization problems with non-linear dynamics in improving the convergence properties by reducing the effort to provide the initial guess. A generalized approach to provide initial guess for the HAM based indirect method makes it more reliable for trajectory optimization.

II. Homotopy Analysis Method Theory & Background

To understand the basic idea of HAM, as developed by Dr. Shijun Liao, we consider an Initial Value Problem (IVP) and extend the discussion to a BVP. Let one of the governing equations be given by a n^{th} order non-linear ODE

$$N[u(r,t),t] = 0, t \in [0,T] \tag{1}$$

subject to n linear initial boundary conditions,

$$B_k[t,u] = \gamma_k, 1 \le k \le n \tag{2}$$

where, N is a n^{th} order differential operator, B_k is a linear operator, u(t) is a smooth function, t is a temporal variable, r is a spatial variable, and $T \ge 0$. For each governing equation N, using an embedding parameter q, Liao suggested to construct a zeroth-order homotopy deformation equation given by Eq. (3), so that the Homotopy-Maclaurin series solution for N, given by $\phi(r,t;q)$ exists and is analytic at q=0. The Homotopy-Maclaurin series solution is shown in Eq. (4).

$$(1-q)L[\phi(r,t;q) - u_0(r,t)] = c_o q H_a N[t, \phi(r,t;q)], c_o \neq 0 \tag{3}$$

$$\phi(r,t;q) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t) q^m, q \in [0,1] \tag{4}$$

L and $u_0(r,t)$ are the linear operator and the initial guess respectively, both provided by the designer. m is the order of Maclaurin series, c_o is an auxiliary convergence control parameter, H_a is a non-zero auxiliary function provided by the designer, and u_m is given by Eq. (5).

$$u_m(r,t) = \frac{1}{m!} \frac{\partial^m \phi(r,t;q)}{\partial q^m}\bigg|_{q=0} \tag{5}$$

From Eq. (3), it can be observed that, as q increases from 0 to 1, the Homotopy-Maclaurin series $\phi(r,t;q)$ varies continuously from $u_0(r,t)$ to $u(r,t)$. We assume that the linear operator, initial guess, auxiliary function and c_o are chosen such that the Maclaurin series converges at $q = 1$. We differentiate the zeroth order deformation equation, Eq. (3) m times with respect to q and divide it by $m!$ to obtain the m^{th} order deformation equation given by Eq. (6). Integrating Eq. (6) with the linear boundary conditions, Eq. (7) gives the HAM series solution, $u(r,t)$ [Eq. (8)]. The value of χ_m is given by Eq. (9).

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = H_a c_o \delta_{m-1}(N[t, \phi(r,t;q)]) \tag{6}$$

$$u_m(r,0) = 0 \tag{7}$$

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t) \tag{8}$$

$$\chi_m = \begin{Bmatrix} 0, m \leq 1, \\ 1, m > 1 \end{Bmatrix} \tag{9}$$

$\delta_k(\phi)$ is called as the k^{th}-order homotopy-derivative operator given in Eq. (10).

$$\delta_k(\phi) = \left(\frac{1}{k!} \frac{\partial^k \phi}{\partial q^k} \right)\bigg|_{q=0} \tag{10}$$

In practice, the series solution obtained by Eq. (8) is truncated to a finite number of terms, which gives us the M^{th} order approximation as

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{M} u_m(r,t) \tag{11}$$

For each value of m=1,2,3..., Eq. (6) converts the non-linear ODE, $N[u(r,t), t] = 0$ to a linear ODE with order equal to the order of the linear operator in use.

II.A. HAM Approach for a Boundary Value Problem

For solving a BVP using HAM, it is first formulated as an IVP. The known initial boundary conditions from Eq. (2) are used in formulating the initial guess as explained below. For the state variables with unknown initial boundary conditions, we assume their values to be finite parameters β_1, β_2,....β_n. After obtaining the series solutions for each state and co-state variable, we use the given terminal boundary conditions to root-solve a non-linear system of n equations to obtain the values of the finite parameters β_1, β_2,....β_n.

The approach is demonstrated with the help of a simple optimal control problem as follows. The objective functional is defined as

$$\text{Min } J = \int_0^1 (x^2 + u^2) dt \tag{12}$$

with the dynamics

$$\dot{x} = u,\ x(0) = 1,\ t \in [0,1] \tag{13}$$

where u is the control variable. On applying the Euler-Lagrange theorem, we obtain the following BVP,

$$\dot{x} = -\lambda, \ \dot{\lambda} = -x, \ x(0) = 1, \ \lambda(1) = 0 \tag{14}$$

where λ is a co-state. This problem has an analytical solution given by Eq. (15)

$$\begin{aligned} x(t) &= \frac{e^t + e^2 e^{-t}}{1 + e^2} \\ \lambda(t) &= -\frac{e^t - e^2 e^{-t}}{1 + e^2} \end{aligned} \tag{15}$$

Homotopy-Maclaurin series for the state and co-state are defined in the following Eq. (16).

$$\begin{aligned} \phi(t; q) &= x_0(t) + \sum_{m=1}^{+\infty} x_m(t) q^m, q \in [0, 1] \\ \psi(t; q) &= \lambda_0(t) + \sum_{m=1}^{+\infty} \lambda_m(t) q^m, q \in [0, 1] \end{aligned} \tag{16}$$

where x_m and λ_m can be obtained by integrating the respective m^{th} order deformation equations given by Eq. (17).

$$\begin{aligned} L[x_m(t) - \chi_m x_{m-1}(t)] &= H_a c_o \delta_{m-1}(N_1[t, \phi(t; q)]) \\ L[\lambda_m(t) - \chi_m \lambda_{m-1}(t)] &= H_a c_o \delta_{m-1}(N_2[t, \psi(t; q)]) \end{aligned} \tag{17}$$

subject to the boundary conditions

$$\begin{aligned} x_m(0) &= 0 \\ \lambda_m(0) &= 0 \end{aligned} \tag{18}$$

where

$$\begin{aligned} N_1 &: \dot{\phi} + \psi = 0 \\ N_2 &: \dot{\psi} + \phi = 0 \end{aligned} \tag{19}$$

M^{th} order HAM solution for the state and co-state variables is represented as

$$\begin{aligned} x(t) &= x_0(t) + \sum_{m=1}^{M} x_m(t) \\ \lambda(t) &= \lambda_0(t) + \sum_{m=1}^{M} \lambda_m(t) \end{aligned} \tag{20}$$

II.B. Selection of Initial Guess, Linear operator, & Auxiliary Function

Although, there are no conclusive proofs and rigorous theories to select the initial guess, linear operator, and the auxiliary function, HAM theory provides suggestions for their selection to solve practical problems.[12] Liao suggested to define a set of basis functions to correctly represent the series solution of the Eq. (1). A typical HAM series solution can be represented as a power series given by

$$u(t) = \sum_{m=1}^{+\infty} a_m e_m(t) \tag{21}$$

where a_m are the coefficients obtained by the HAM series solution, and $e_m(t)$ are the basis functions, chosen by the designer to represent the series solution. Eq. (21) is called as the *rule of solution expression*. The initial guess, linear operator, and the auxiliary function has to be chosen in such a way so that they satisfy the *rule of solution expression* as explained below.

II.B.1. Initial Guess

The initial guess must be chosen such that it can be expressed by the sum of the basis functions defined above. Also, the initial guess for a state must satisfy the maximum possible number of boundary conditions for that state. Eq. (22) shows a typical representation of the initial guess for a state

$$x_0(t) = \sum_{m=1}^{n} b_m e_m(t) \tag{22}$$

where n is the number of boundary conditions on the state, b_m are the finite coefficients chosen by the designer to satisfy the boundary conditions, and $e_m(t)$ are the basis functions chosen to represent the series solution.

II.B.2. Linear Operator

The linear operator must be chosen such that the solution of Eq. (23) is expressed as the sum of the basis functions chosen earlier and is given by Eq. (24)

$$L[w(t)] = 0 \tag{23}$$

$$w(t) = \sum_{m=1}^{K_1} d_m e_m(t) \tag{24}$$

where, d_m are a set of finite coefficients chosen by the designer and K_1 is a positive integer. There is no strict rule to select the value of K_1, but it is suggested that in most of the problems, it should be chosen as the highest order of derivative of the original Eq. (1). For finite time interval BVPs, where $t \in [0, T]$, H_a is simply used as 1.

For the current problem we use the simplest *rule of solution expression*, a polynomial power series given by Eq. (26) for which the set of basis functions is the following set e_m

$$e_m(t) = [1, t, t^2, t^3....] \tag{25}$$

$$x(t) = a_1 + a_2 t + a_3 t^2 + ... \tag{26}$$

where $a_1, a_2,$ and a_3 are the coefficients which are obtained by the HAM series solution. In general, we have the freedom to choose the basis functions as polynomials, trigonometric functions, Fourier series or a combination of them.

The approach mentioned in Section II.B.1 is used to select the initial guesses for the state and co-state. For convenience, we decide to select the initial guesses which satisfy only the initial boundary conditions for both the state and the co-state. This assumption results in $n = 1$ for the initial guess. Hence,

$$\begin{aligned} x_0(t) &= \sum_{m=1}^{1} b_{1m} e_m(t) = b_{11} e_1 \\ \lambda_0(t) &= \sum_{m=1}^{1} b_{2m} e_m(t) = b_{21} e_1 \end{aligned} \tag{27}$$

To satisfy the initial boundary condition on the state, we select $b_{11} = 1$. Since the initial boundary condition for the co-state is unknown, we assume it to be some finite value β_1 as explained in Section II.A. We select $b_{21} = \beta_1$ to satisfy the initial boundary condition on the co-state. The initial guesses for the state and co-state are now calculated to be:

$$\begin{aligned} x_0(t) &= 1 \\ \lambda_0(t) &= \beta_1 \end{aligned} \tag{28}$$

For the linear operator, we use the approach described in Section II.B.2 to define w(t) as shown in Eq. (29). Since the highest order derivative for both the original governing equations is 1, the value for K_1 is chosen to be 1. We can now define w(t) as

$$w(t) = \sum_{m=1}^{1} d_m e_m = d_1 e_1 = d_1 \tag{29}$$

where, d_1 is a constant. The linear operator is chosen to be a first order derivative, such that it satisfies Eq. (23) as follows:

$$L(w(t)) = \frac{d}{dt}(d_1) = 0 \tag{30}$$

Using Eq. (17) and Eq. (19), the m^{th} order deformation equations can now be written for the state and costate as

$$\begin{aligned} x_m(t;c_o) &= \chi_m x_{m-1}(t) + c_o \int_0^t H_a \delta_{m-1}(\dot{x} + \lambda)dt + C_1 \\ \lambda_m(t,c_o) &= \chi_m \lambda_{m-1}(t) + c_o \int_0^t H_a \delta_{m-1}(\dot{\lambda} + x)dt + C_2 \end{aligned} \qquad m\text{=1,2,3,...} \tag{31}$$

C_1 and C_2 are constants of integration determined by the initial conditions given by Eq. (18). As mentioned previously, we assume $H_a = 1$.

II.C. Auxiliary Convergence Control Parameter (c_o)

HAM guarantees the convergence of the series solution,[12] which counts as one of the major advantages to solve BVPs. MATLAB's symbolic toolbox is used to solve Eq. (31). We obtain terms for x_m and λ_m and substitute them into Eq. (20), to obtain a family of series solutions in c_o. The solutions for state and co-state are functions of the independent variable t and c_o. The Homotopy Analysis Method provides us the freedom to choose the value of the c_o to adjust the region and the rate of convergence. Liao suggested to plot the curves of physical quantities like $\dot{x}|_{t=t'}$, $\ddot{x}|_{t=t'}$ with c_o to study their dependency on c_o, where t' can be any instant of time in the domain of the problem. These curves are termed as $c_o \sim$ curves and are denoted as $\dot{x} \sim c_o$ and $\ddot{x} \sim c_o$ for any state or physical quantity. According to the HAM convergence theorem,[12] all convergent series of $\dot{x}$ and $\ddot{x}$ converge to constant values for a specific range (R_{c_o}) of c_o values, resulting in a horizontal line in the $c_o \sim$ curves. Regardless of the initial guess, and for any value of c_o in R_{c_o}, the same value of the physical quantity is obtained, and the series solution is said to converge.

II.D. Discrete Squared Residual (E_{md})

Squared residual is defined as a measure of how well the power series satisfies the governing equations integrated over the whole domain. The squared residual for any governing equation is calculated as

$$E_m(c_o) = \int_0^T \left\{ N[\textstyle\sum_{n=1}^{m} u_m(t,c_o)] \right\}^2 dt \tag{32}$$

where T is the final value of time interval used in Eq. (1), E_m is the squared residual for the governing equation, obtained at m^{th} order series solution. c_o plays an important role in determining the residual for any series solution. As proposed by Liao,[11] once R_{c_o} is determined, the optimal value of c_o can be calculated by minimizing the squared residual within R_{c_o} .
Due to the high computational requirements for E_m, a discrete squared residual E_{md}, is also defined for the m^{th} order series solution as

$$E_{md} = \frac{1}{N_{step}+1} \sum_{j=0}^{N_{step}} \left\{ \Delta_m(\tau_j;c_o) \right\}^2, \ \tau_j = \frac{t_f j}{N_{step}} \tag{33}$$

where

$$\Delta_m(\tau;c_o) = N(u_m(\tau;c_o)) \tag{34}$$

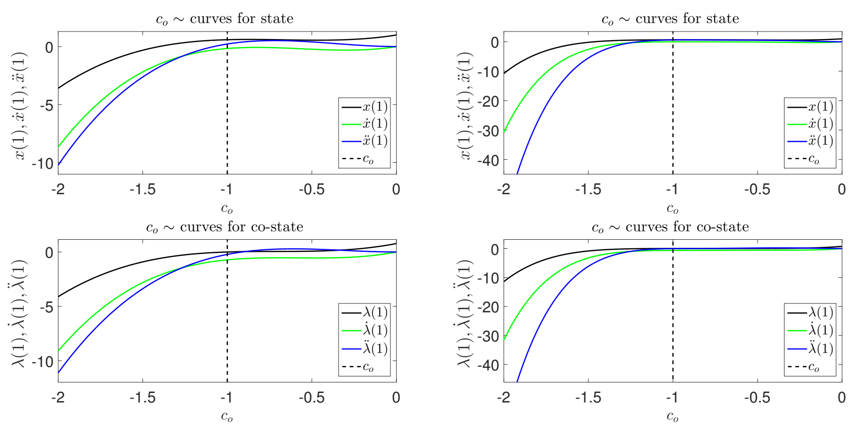
N_{step} is the number of time steps used, and t_f is the final time of the OCP. For the current study, N_{step} is assumed to be 40. An overall discrete squared residual $E_{md,total}$ can be defined by adding the discrete squared residuals for each governing equation as follows

$$E_{md,Total} = E_{md,N_1} + E_{md,N_2} + E_{md,N_3} + ... \tag{35}$$

where E_{md,N_1} is the discrete squared residual for the governing equation N_1.

The m^{th} order deformation Eq. (31) is solved to obtain the analytical terms for x_m and λ_m terms in c_o and β_1. First, the value of c_o is assumed to be -1. Then, the final boundary condition on the co-state is used to obtain a non-linear equation which is root solved for β_1. Using β_1, the M^{th} order series solutions for both the state and co-state are evaluated.

Further, the $c_o \sim$ curves are plotted to understand the convergence properties of the state and co-state series solutions. For this problem, the quantities $x \sim c_o$, $\dot{x} \sim c_o$, $\ddot{x} \sim c_o$ for the state and $\lambda \sim c_o$, $\dot{\lambda} \sim c_o$, $\ddot{\lambda} \sim c_o$ for the co-state are used. Theoretically, since the curves converge at each instant of time, the final time of 1 s is chosen arbitrarily. Figs. 1(a) and 1(b) show the $c_o \sim$ curves for the 3^{rd} order and 5^{th} order series solutions respectively. The two plots can be compared to conclude that the convergence region increases with an increase in the order of solution, giving designers more freedom in choice of c_o. For the 5^{th} order solution, a common c_o range of [-1.2, 0] could be identified for both the state and the co-state in which the curves converge to constant values for all the mentioned physical quantities.



(a) 3^{rd} order $c_o \sim$ curve (b) 5^{th} order $c_o \sim$ curve

Figure 1: $c_o \sim$ Curves for Simple Optimal Control Problem

$E_{md,Total}$ given by Eq. (35), is minimized to obtain the optimal c_o for the 5^{th} order solution. MATLAB's *fminbnd* function based on the Golden Section Search Algorithm with parabolic interpolation is used to minimize $E_{md,total}$ for the range [-1.2, 0]. The optimal value of c_o for the 5^{th} order solution is found to be -0.9567. The optimal c_o is then used to obtain the updated series solutions for the state and co-state. β_1 is root solved again by using the terminal boundary condition on the co-state. This method of using an updated value of c_o to obtain the series solutions is known as "convergence control".
Table 1 shows a reduction in the total discrete squared residual obtained by using an optimal value of c_o, as compared to c_o of -1. The computations were performed on an Intel(R) Xeon(R) CPU-E3-1225 v3 3.20 Ghz (4 CPUs) processor. Fig. 2 shows the performance curve for the problem. Reduction in the discrete squared residual is traded with increase in the computational cost. It is found that the residual decreases with increase in the order of series solution.

Table 1: Effect of c_o on $E_{md,Total}$ - Simple Optimal Control Problem

c_o	$E_{md,Total}$
-1	3.17×10^{-5}
-0.9567	1.27×10^{-6}

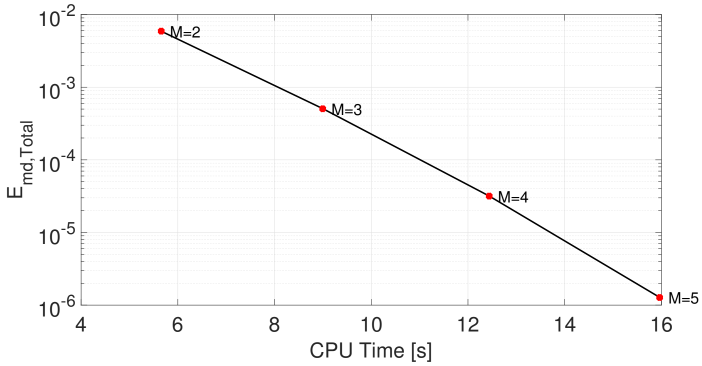


Figure 2: Performance curve for Simple Optimal Control Problem

The series solutions for both the state and costate are compared with the analytical solution in the Fig. 3. The series solutions without convergence control ($c_o = -1$) are also compared with the solutions obtained using the optimal c_o. Although, both of the values of c_o lie in the convergence region $[R_{c_o}]$, a small improvement is obtained by using the optimal value of c_o. The 5^{th} order series solution is given by Eq. (36).

$$\begin{aligned} x(t) &= 1 - 0.76t + 0.5t^2 - 0.12t^3 + 0.04t^4 - 0.01t^5 \\ \lambda(t) &= 0.76 - t + 0.38t^2 - 0.16t^3 + 0.03t^4 - 0.01t^5 \end{aligned} \tag{36}$$

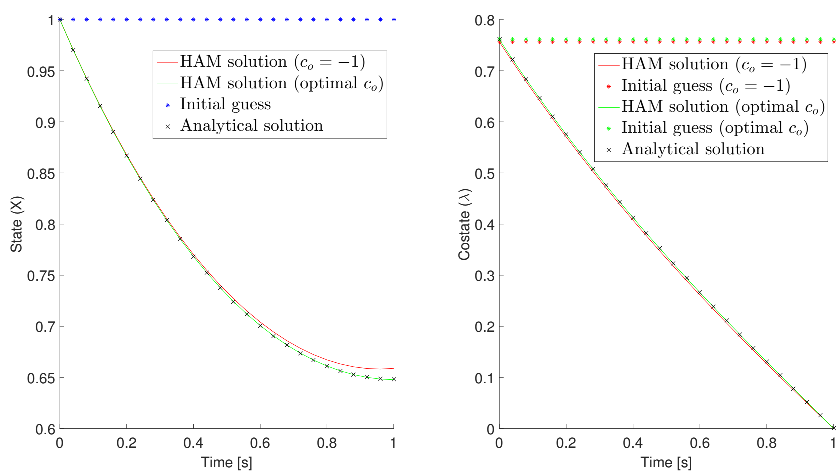


Figure 3: State and Co-state 5^{th} order HAM solution for the Simple Optimal Control Problem.

Different initial guesses were used to compute the HAM series solution. Table 2 shows the $E_{md,Total}$ values obtained at 5^{th} order HAM solution for the initial guesses used. The initial guess given in the last row is based on an exponential *rule of solution expression*. Since the analytical solution of the problem (Eq. (15)) contains exponential terms, the exponential series *rule of solution expression* is an apt choice for this particular problem. This fact is also confirmed by the least value of $E_{md,Total}$ obtained by using the exponential initial guess as compared to other expressions.

Table 2: Effect of Initial Guess on $E_{md,Total}$

Initial Guess $[x_0, \lambda_0]$	$E_{md,Total}$
$[1, \beta_1]$	1.27×10^{-6}
$[1, \beta_1(1-t)]$	1.90×10^{-6}
$[1, \beta_1 e^t]$	1.37×10^{-5}
$[e^t, \beta_1 e^t]$	1.41×10^{-8}

III. Zermelo's Problem

Zermelo's problem[13] consists of minimizing the time required by a boat to cross a river. Fig. 4 shows a schematic of the optimal control problem. θ is the boat steering angle from the horizontal direction which is varied continuously to reach the terminal point across the river in the minimum possible time. The boat is assumed to move with a constant velocity, V of 1 m/s.

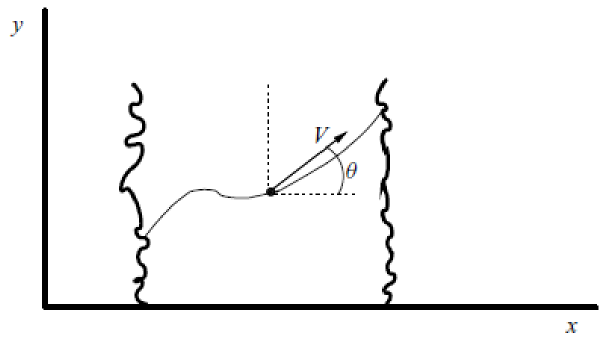


Figure 4: Schematic for Zermelo's problem. Figure in citation.[13]

The objective function, J, is defined as

$$\text{Min } J = t_f \tag{37}$$

with the dynamics

$$\begin{aligned} \dot{x} &= V\cos\theta \\ \dot{y} &= V\sin\theta \end{aligned} \tag{38}$$

where, the states x and y are the horizontal and vertical coordinates respectively. We use the Euler-Lagrange theorem to obtain the dynamics for the co-states given in Eq. (39). The control law is evaluated and is given by Eq. (40).

$$\begin{aligned} \dot{\lambda}_x &= 0 \\ \dot{\lambda}_y &= 0 \end{aligned} \tag{39}$$

$$\tan\theta = \frac{\lambda_y}{\lambda_x} \tag{40}$$

The boat starts from (0,0) and crosses the river to reach (1,1). The set of boundary conditions along with the transversality condition obtained for the final time are given in Eq. (41).

$$\begin{aligned} x(0) = 0,\ y(0) &= 0 \\ x(t_f) = 1,\ y(t_f) &= 1 \\ H_{t_f} &= -1 \end{aligned} \tag{41}$$

where H_{t_f} is the value of the hamiltonian calculated at the final time t_f. This problem has an analytical solution given by Eq. (42).

$$
\begin{aligned}
x(t) &= \frac{1}{\sqrt{2}}t \\
y(t) &= \frac{1}{\sqrt{2}}t \\
t_f &= \frac{1}{\sqrt{2}}s \\
\theta(t) &= 45^o \\
\lambda_x(t) &= \lambda_y(t)
\end{aligned} \tag{42}
$$

III.A. HAM Problem Formulation

The Homotopy-Maclaurin series and the m^{th} order deformation equations for the states and co-states are formulated. A set of polynomial functions as the basis functions and a *rule of solution expression* similar to the one used in the previous problem given by Eq. (26) is chosen for Zermelo's problem. Similar to the process before, for convenience, we select the initial guesses which satisfy only the initial boundary conditions on the states and co-states. The initial guess constructed is given in the Table 3.

Table 3: Initial Guess for Zermelo's Problem

State/Co-state	Initial Guess
x	0
y	0
λ_x	β_1
λ_y	β_2

Since the highest order of derivative for all the governing equations is 1, we select the value of K_1 defined in Eq. (24) to be 1. Similar to the previous problem, the linear operator is calculated to be $\frac{d}{dt}$. The auxiliary function is chosen to be 1. Since we are also required to calculate the optimal final time for the problem, another unknown parameter, t_f, is used. We start by selecting the value of c_o to be -1. The terminal boundary conditions on the states and the transversality condition on the final time parameter, t_f, are applied to formulate a non-linear system of equations in the parameters β_1, β_2, and t_f. MATLAB's Fsolve function is used to numerically solve the non-linear system of equations.

III.B. Results for Zermelo's Problem

Fig. 5 shows the $c_o \sim$ curves for both the state variables at the 5^{th} order solution for $c_o \in [-2, 0]$. Since the co-state values are constant, they are not dependent on c_o. For the states, $c_o = -1$ lies in the horizontal convergence region, and can be used to evaluate the series solutions. Hence,"convergence control" is not required for this problem.

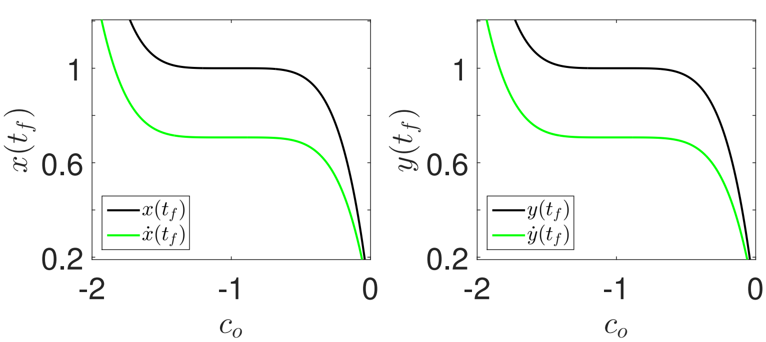


Figure 5: $c_o \sim$ curves for Zermelo's problem: $c_o \in [-2, 0]$

The series solutions for the states and co-states are compared with the analytical results in Fig. 6. State variable trajectories and t_f calculated from the HAM approach agree with the analytical solution. Co-state variables are constant and can take any non-zero real value. Fig. 7(a) compares the control history obtained using the two methods. A constant steering angle of 45 deg is needed to be maintained for the boat to reach the terminal point. Fig. 7(b) shows that a first order HAM solution results in a $E_{md,Total}$ of the order of 10^{-26}.

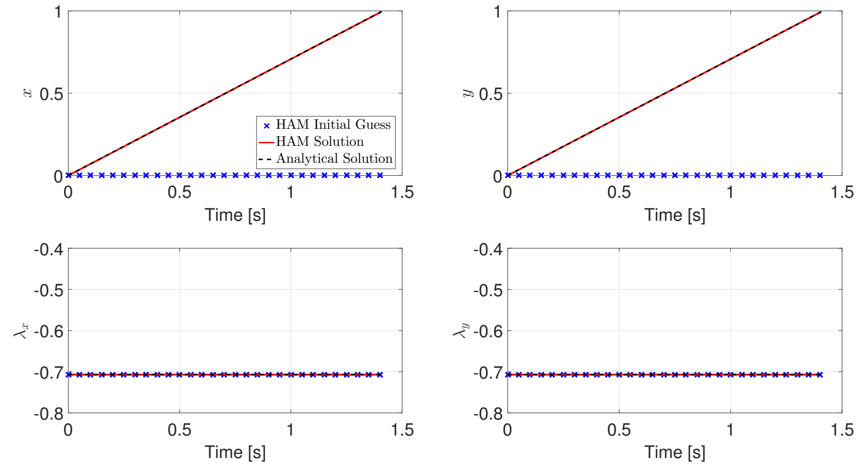
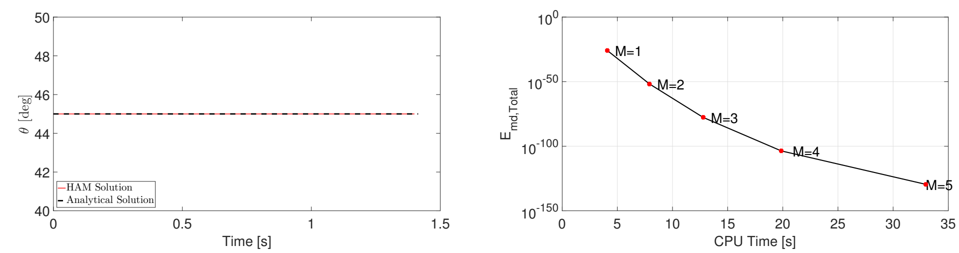


Figure 6: State and Co-state - 5^{th} order HAM solution for Zermelo's Problem



(a) Control history - Zermelo's Problem.

(b) Performance Curve - Zermelo's Problem

Figure 7: Results - Zermelo's Problem

IV. 2D Ascent Problem

A finite time 2D ascent problem is solved to demonstrate the applicability of the HAM based indirect method on a non-linear aerospace trajectory optimization problem. The 2D ascent problem is a popular optimal control problem emulating the satellite launch from the surface of the Earth to an orbit of fixed altitude in the minimum possible time.

IV.A. Problem Formulation

A modified 2D ascent problem has been constructed to test the HAM approach on the optimal control problem. A flat Earth model is assumed as shown in Fig. 8. The objective of this optimal control problem is to maximize the final horizontal velocity of the vehicle in orbit in a given fixed time. The original 2D ascent problem[13] has been simplified to a fixed final time problem by using the following assumptions:

1. A vehicle launched from the surface of the Earth must reach an orbit of 185.2 km in 485 seconds to achieve a maximum terminal horizontal component of velocity.

2. An instantaneous steering angle α is the only control variable.

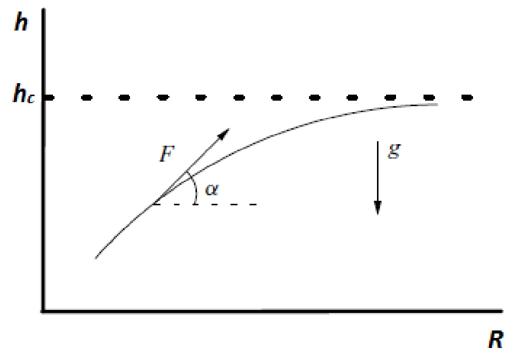


Figure 8: Flat Earth model for 2D ascent problem.[13] Figure in citation

3. The acceleration due to gravity from the Earth is assumed to be 9.8 m/s^2.
4. Thrust to weight ratio for the vehicle is 3.
5. A constant mass and a constant thrust force F is assumed.
6. Final altitude to be achieved is 185.2 km.
7. There is no atmosphere and no aerodynamic forces on the vehicle.

The objective function for this case is defined by Eq. (43)

$$\text{Min } J = \; -v_{x_f} \tag{43}$$

with the dynamics given by Eq. (44). The thrust acceleration for the vehicle is calculated using Eq. (45)

$$\begin{aligned} \dot{R} &= v_x \\ \dot{h} &= v_y \\ \dot{v_x} &= \frac{F}{m_o}\cos\alpha \\ \dot{v_y} &= \frac{F}{m_o}\sin\alpha - g \end{aligned} \tag{44}$$

$$\frac{F}{m_o} = (\text{Thrust to weight})(g) = 29.4\,\text{m/s}^2 \tag{45}$$

The boundary conditions on the states and co-states for the boundary value problem formulated are given in Table. 4.

Table 4: Boundary Conditions - 2D ascent problem.

State/Co-state	Initial Condition	Terminal Condition
R	0 km	free
h	0 km	185.2 km
v_x	0 km/s	free
v_y	0 km/s	0 km/s
λ_R	free	0
λ_h	free	free
λ_{vx}	free	-1
λ_{vy}	free	free

IV.B. HAM Formulation

Following the similar approach as described in the Section II.A, we formulate the HAM problem by defining the Homotopy-Maclaurin series and the mth order deformation equations for each state and co-state variable.

The choice of the basis functions and the *rule of solution expression* is the same as used for the simple control problem given by Eq. (25). Same approach is used for the selection of initial guess as used for the previous problems. Only the initial boundary conditions are considered to build the initial guess, as shown in the Table 5. From Table 4, it can be seen that the initial conditions for the co-states are unknown. Hence, we select parameters for those values.

Table 5: Initial Guess - 2D Ascent Problem

State and Co-state	Initial Guess
R	0
h	0
v_x	0
v_y	0
λ_R	β_1
λ_h	β_2
λ_{v_x}	β_3
λ_{v_y}	β_4

Same approach for the selection of linear operator and auxiliary function are used as described in the Section II.B. The value of K_1 is chosen to be 1, since the highest order of derivative for all the governing equations is 1. Using Eq. (23), we obtain the linear operator as $\frac{d}{dt}$. The auxiliary function is selected to be 1. The 4 terminal boundary conditions given in Table 4 are used to build a nonlinear system of equations in β_1, β_2, β_3 and β_4. Table 6 lists the values of parameters obtained at each order of solution.

Table 6: Parameter Values - 2D Ascent Problem

Order of solution	β_1	β_2	β_3	β_4
2	0	-2.3230$\times10^{-4}$	-1	-0.4144
3	0.0008	-2.3230$\times10^{-4}$	-1	-0.5975
4	3.7368$\times10^{-30}$	-6.2122$\times10^{-4}$	-1	-0.5133
5	1.1183$\times10^{-4}$	-0.0011	-1	-0.6333
6	4.8850$\times10^{-31}$	-7.4772$\times10^{-4}$	-1	-0.5423
7	-7.0779$\times10^{-29}$	-8.3634$\times10^{-4}$	-1	-0.5611
8	-5.7842$\times10^{-31}$	-7.8445$\times10^{-4}$	-1	-0.5501
9	-4.4299$\times10^{-30}$	-8.0683$\times10^{-4}$	-1	-0.5548
10	-7.9876$\times10^{-16}$	-7.9460$\times10^{-4}$	-1	-0.5522
10th order for optimal c_o	6.9437$\times10^{-29}$	-0.0007	-1	-0.5522

IV.C. Results for 2D Ascent Problem

Fig. 9 shows the c_o curves for the 10th order series solutions. No common horizontal region can be found out for which all the physical quantity converge. In this case, optimal c_o is calculated to be -0.9996 by simply minimizing $E_{md,Total}$ as a function of c_o for real values. Figs. 10 and 11 compare the state and co-state series solution with the *bvp4c* solution. It can be seen from Fig. (10) that an initial guess of 0 is deformed into a highly non-linear solution using this approach. Fig. 12 compares the the control history obtained from the two methods. Performance curve for the ascent problem is shown by Fig. 13. An irregular peak for 5th order solution is due to inability of the MATLAB's numerical solver to obtain the solution in 2,000,000 function evaluations. Since the number of analytical terms in the deformation equations increase with the order, the CPU time for the 10th order solution almost triples to that of the 9th order solution.

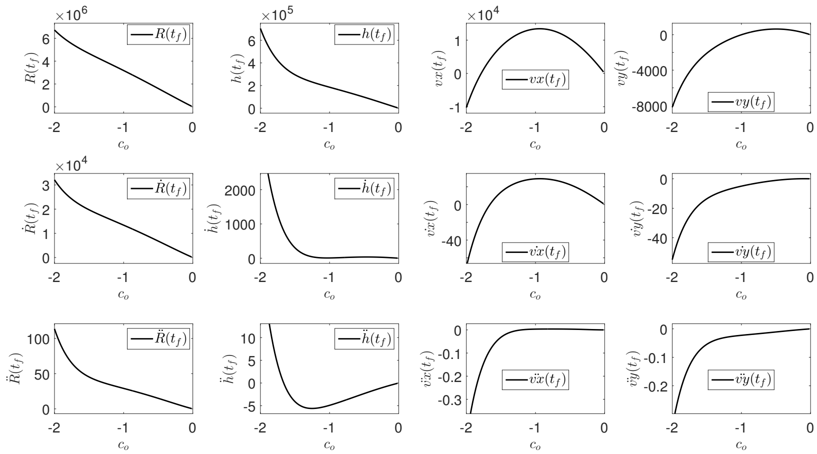


Figure 9: $c_o \sim$ curves for $c_o \in [-2, 0]$ - 2D Ascent Problem

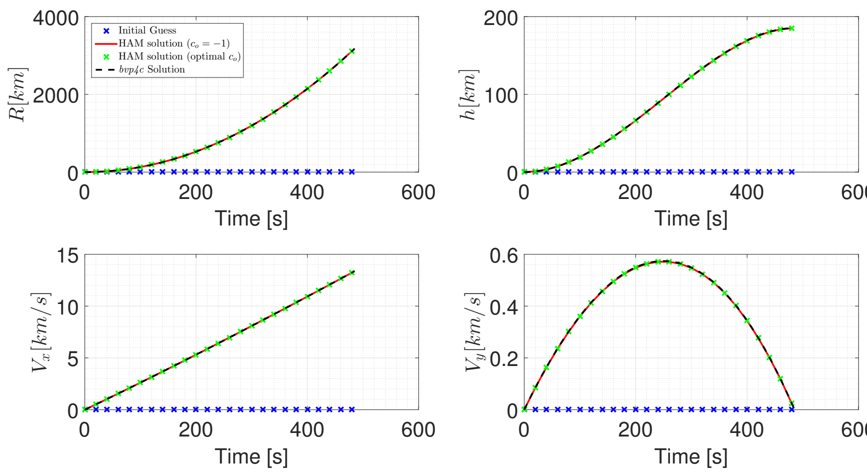


Figure 10: States for 10^{th} Order HAM Solution - 2D Ascent Problem

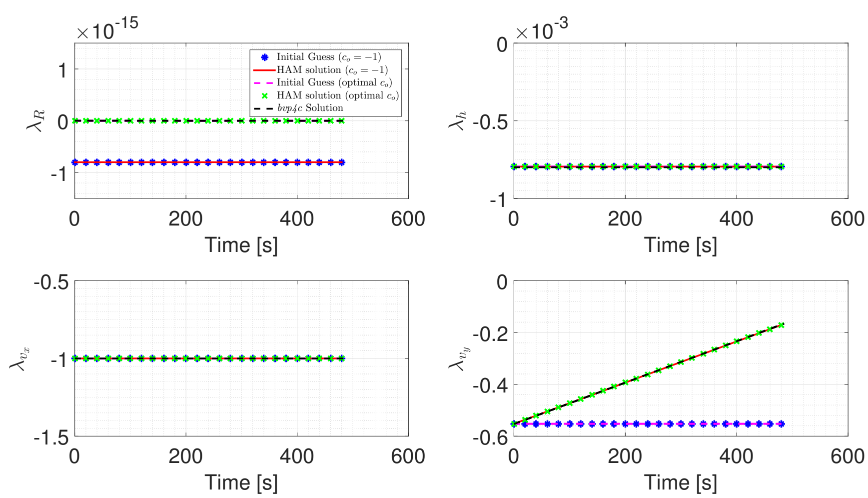


Figure 11: Co-states for 10^{th} Order HAM solution - 2D Ascent Problem

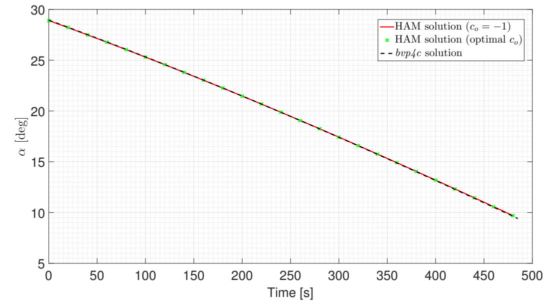


Figure 12: Control History - 2D Ascent Problem

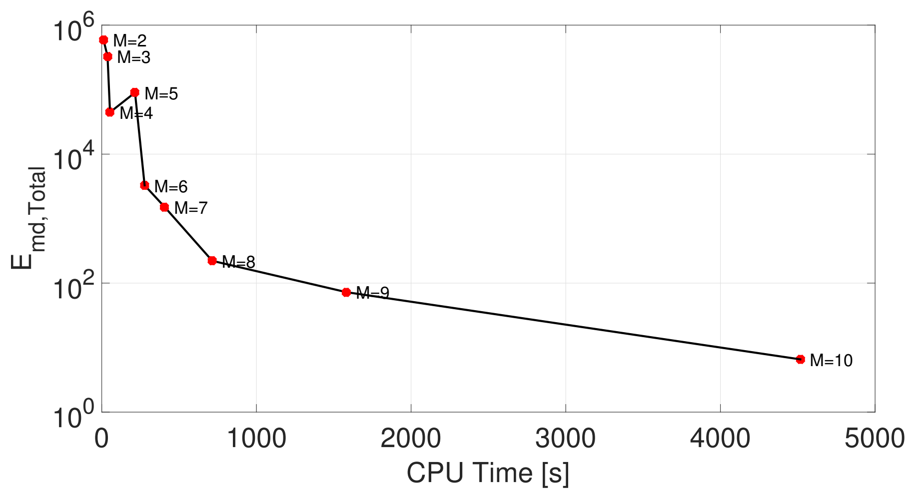


Figure 13: Performance Curve - 2D Ascent Problem

V. Conclusion and Future Work

HAM based indirect method for trajectory optimization is demonstrated on linear and non-linear optimal control problems. The HAM based solver exploits the flexibility and ease available for the initial guess as compared to the collocation solver. Use of convergence control in a region is also demonstrated in the approach. The HAM solver currently suffers from high computational times which can be further worked upon. This study concludes by showing an immense potential in the development of HAM based indirect solvers for trajectory optimization. Further work in this area is mentioned as follows

1. Process of integrating the symbolic deformation equations can be parallelized to reduce computational time.
2. Quantifying the rate of convergence can be done to observe the process of convergence control. This will further help in classifying optimal control problems based on their convergence properties.
3. As suggested in the theory, multiple convergence control parameters can be used further to control the rate of convergence.
4. For higher fidelity solutions, the approach can be used to build hybrid solvers with faster solvers like collocation and pseudospectral methods.

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